International Journal of Theoretical Physics, Vol. 44, No. 4, April 2005 (© 2005)
DOI: 10.1007/s10773-005-3977-6

Remarks on Perturbation of Infinite Networks of Identical Resistors

J. H. Asad,[1] R. S. Hijjawi,[2] A. Sakaj,[3] and J. M. Khalifeh[1,4]

Received July 31, 2004; accepted October 12, 2004

The resistance between arbitrary sites of infinite square network of identical resistors is studied when the network is perturbed by removing two bonds from the perfect lattice. A connection is made between the resistance and the lattice Green's function of the perturbed network. By solving Dyson's equation the Green's function and the resistance of the perturbed lattice are expressed in terms of those of the perfect lattice. Some numerical results are presented for an infinite square lattice.

KEY WORDS: Lattice Green's function; resistors; perturbation.

1. INTRODUCTION

A classic problem in electric circuit theory studied by many authors over many years is computation of the resistance between two nodes in a resistor network. Besides being a central problem in electric circuit theory, the computation of resistances is also relevant to a wide range of problems ranging from random walk (Doyle and Snell, 1984; Lovăz, 1996), theory of harmonic functions (Van der Pol and Bremmer, 1955) to first-passage processes (Redner, 2001).

The connection with these problems originates from the fact that electrical potentials on a grid are governed by the same difference equations as those occurring in the other problems. For this reason, the resistance problem is often studied from the point of view of solving the difference equations, which is most conveniently carried out for infinite networks.

Kirchhoff (1847) formulated the study of electric networks more than 150 years ago, and the electric–circuit theory has been discussed in detail by

[1] Department of Physics, University of Jordan, Amman-11942, Jordan; e-mail: jihadasd2002@yahoo.com.
[2] Department of Physics, Mutah University, Jordan.
[3] Department of Physics, Ajman University, UAE; e-mail: info@ajss.net.
[4] To whom correspondence should be addressed at Department of Physics, University of Jordan, Amman-11942, Jordan; e-mail: jkalifa@ju.edu.jo.

Van der Pol and Bremmer (1955) in which they derived the resistance between nearby points on the square lattice. Bartis (1967) introduced how complex systems can be treated at the elementary level and showed how to calculate the effective resistance between adjacent nodes of a square, triangular, honeycomb and kagome lattices of one-ohm resistors.

Venezian (1994) showed that the resistance between adjacent sites on an infinite square grid of equal resistors can easily be found by the superposition of current distribution. The mathematical problem involves the solution of an infinite set of linear, inhomogeneous difference equations which are solved by the method of separation of variables. Numerical results for the resistances between the sites (0, 0) and (l, m) in units of R are presented. Atkinson and Van Steenwijk (1999) calculated the resistance between two arbitrary sites in an infinite square lattice of identical resistors. Their method is generalized to infinite triangular- and hexagonal-lattices in two dimensions, and also to infinite cubic and hypercubic-lattices in three and more dimensions.

Monwhea (2000) introduced a mapping between random walk problems and resistor network problems, where his method was used to calculate the effective resistance between any two sites in an infinite two-dimensional square lattice of unit resistors and the superposition principle was used to find the effective resistances on toroidal- and cylindrical–square-lattices.

Recently, Cserti (2000) introduced an alternative method based on the LGF rather than using the superposition distribution of current, where the resistance for d-dimensional hypercubic- rectangular- triangular- and honeycomb-lattices of resistors is discussed in detail. Recurrence formulae for the resistance between arbitrary lattice points of the square lattice have been given in his paper. The resistance between arbitrary nodes of infinite networks of resistors is studied when the network is perturbed by removing one bond from the perfect lattice (Cserti *et al.*, 2002), where the resistance in a perturbed lattice is expressed in terms of the resistance in a perfect lattice. Wu (2004) studied the finite networks consisting of identical resistors using the so-called Laplacian matrix where he obtained the resistance between two arbitrary lattice sites in terms of the eigenvalues and eigenfunctions of the Laplacian. Wu's obtained explicit formulae for the resistance in one, two and three dimensions under various boundary conditions. Finally, Asad *et al*. (2004) studied both the perfect and perturbed infinite square mesh using Cseti's method where they construct a finite network consisting of (30×30) identical resistance. They obtained measurements for the resistance between the origin and other lattice sites and they compared these measured values with those obtained by Cserti's method.

The properties of the Lattice Green's Function (LGF) have been studied in details (Morita and Horiguchi, 1972), especially when impurities are often introduced. The LGF for square lattice has been studied well by many authors (Hijjawi, 2002; Morita and Horiguchi, 1971; Morita, 1971), the LGF for the rectangular

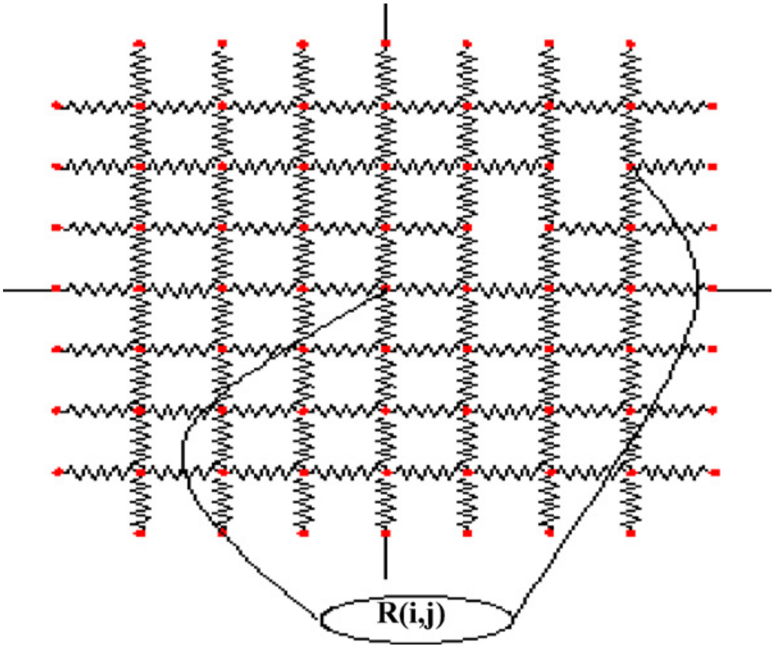

Fig. 1. Perturbation of an infinite square lattice by removing two edges between sites $(i_0 j_0)$ and $(k_0 l_0)$. The resistance $R(i, j)$ between arbitrary lattice points i and j.

lattice has been investigated (Katsura and Inawashiro, 1971). Recurrence relation, which gives the LGF along the diagonal direction from a couple of values of complete elliptic integrals of the first and second kinds for the rectangular and square lattices, has been derived (Morita, 1971), and in these references the reader can find useful papers.

In this work, we study the perturbation of infinite networks when two bonds are broken. As an example (see Fig. 1), consider an infinite square lattice whose edges represent identical resistances R removing two edges (bonds) from this perfect lattice results in a perturbed lattice.

2. PERFECT CASE

Before starting with the formalism of the perturbed lattice, let us first review the perfect case using Dirac's notation. To do this, consider a perfect d-dimensional infinite lattice made of identical resistances R, where all lattice points are specified by the poison vector $\vec{r}$ defined as

$$\vec{r} = l_1\vec{a}_1 + l_2\vec{a}_2 + \cdots + l_d\vec{a}_d. \tag{1}$$

The potential at site r_i be $V(r_i)$, the current entering at origin to be $(+I)$, and the current exiting at a lattice point r_i to be $(-I)$. One can form two vectors, V and I such that

$$V = \sum_i |i\rangle V_i. \tag{2}$$

$$I = \sum_i |i\rangle I_i. \tag{3}$$

where

$$V_i = V(r_i);$$

and

$$I_i = I(r_i).$$

We assume that $\langle i|k\rangle = \delta_{ik}$ and $\sum_i |i\rangle\langle i| = 1$. (i.e. $|i\rangle$ forms a complete orthonormal set).

Using Eqs. (1) and (2), then according to Ohm's and Kirchhoff's laws one gets (Cserti *et al.*, 2002)

$$\sum_j (z\delta_{ij} - \Delta_{ij})\langle j|V = R\langle i|I. \tag{4}$$

Multiplying Eq. (4) by $|i\rangle$ and taking the sum over i, one gets

$$L_0 V = -IR \tag{5}$$

where $L_0 = \sum_{i,j} |i\rangle(\Delta_{ij} - z\delta_{ij})\langle j|$ is the so-called lattice Laplacian.

$z \equiv$ no. of nearest neighbors of each lattice site (i.e., $z = 2d$).

$$\Delta_{kl} = \begin{cases} 1, & r_k, r_l \text{ nearest neighbors} \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

The Lattice Green's Function (LGF) is defined by Economou (1983) as

$$L_0 G_0 = -1. \tag{7}$$

The solution of Eq. (4) in its simplest form is

$$V = -R_0 L_0^{-1} I = R G_0 I_m. \tag{8}$$

Assume a current $+I$ enters at site $\vec{r}_i$,$-I$ exits at site $\vec{r}_j$ and zero otherwise. Thus, the current distribution may be written as

$$I_m = I(\delta_{mi} - \delta_{mj}), \quad \forall m. \tag{9}$$

Substituting Eq. (8) into Eq. (7), one gets

$$V_k = \langle k|V = \sum_m \langle k|RG_0 I_m|m\rangle = RI[G_0(k,i) - G_0(k,j)]. \tag{10}$$

The resistance between the sites r_i and r_j is

$$R_0(i,j) = \frac{V_i - V_j}{R} \tag{11}$$

Using Eq. (9) then Eq. (10) becomes

$$R_0(i,j) = 2R[G_0(i,i) - G_0(i,j)] \tag{12}$$

where we have made use of the symmetry properties of the LGF.

To study the asymptotic behavior of the resistance in an infinite square lattice, it has been shown (Cserti *et al.*, 2002) that for large separation between the two sites *i* and *j* the resistance becomes:

$$\frac{R_0(i,j)}{R} = \frac{1}{\pi}\left[Ln\sqrt{(j_x - i_x) + (j_y - i_y)} + \gamma + \frac{Ln8}{2}\right]. \tag{13}$$

Thus, as the separation between the two sites *i* and *j* goes to infinity the perfect resistance in an infinite square lattice goes to infinity, the same result has been showed in Doyle's and Snell's book (Doyle and Snell, 1984) without deriving the asymptotic form.

3. PERTURBED CASE (TWO BONDS ARE BROKEN)

The current contribution δI_{i1} at the site r_i due to the bond $(i_0 j_0)$ is given by

$$\begin{aligned}\delta I_{i1}R &= \delta_{ii_0}(V_{i_0} - V_{j_0}) + \delta_{ij_0}(V_{j_0} - V_{i_0}) = \langle i|i_0\rangle(\langle i_0| - \langle j_0|)V \\ &\quad + \langle i|j_0\rangle(\langle j_0| - \langle i_0|)V = \langle i|(|i_0\rangle - |j_0\rangle)(\langle i_0| - \langle j_0|)V\delta I_{i1}R = \langle i|L_1 V.\end{aligned} \tag{14}$$

where the operator L_1 is of a so-called "dyadic" form

$$L_1 = (|i_0\rangle - |j_0\rangle)(\langle i_0| - \langle j_0|)$$

and $\langle n|m\rangle = \delta_{nm}$ has been used.

Replacing the bond $(i_0 j_0)$ by $(k_0 l_0)$ then, the current contribution δI_{i2} at the site r_i due to the bond $(k_0 l_0)$ is given by

$$\delta I_{i2}R = \langle i|L_2 V. \tag{15}$$

where the operator L_2 has the form

$$L_2 = (|k_0\rangle - |l_0\rangle)(\langle k_0| - \langle l_0|). \tag{16}$$

Now removing the bonds $(i_0 j_0)$ and $(k_0 l_0)$ from the perfect lattice, then the current I_i at the site r_i is given by

$$(-L_0 V)_i - R\delta_{i1} - R\delta_{i2} = RI_i \tag{17}$$

or

$$LV = -RI_i \tag{18}$$

where

$$L = L_{01} + L \quad \text{and} \quad L_{01} = L_0 + L_1.$$

The LGF for the perturbed lattice can be written as

$$LG = -1 \tag{19}$$

To measure the resistance between r_i and r_j, we assume the current distribution to be as given in Eq. (8).

Now, the simplest solution of Eq. (17) is

$$V = -RI_i L^{-1} \tag{20}$$

From Eq. (19) $L^{-1} = -G$. Thus, Eq. (20) becomes

$$V = RI_i G \tag{21}$$

To obtain the potentials at different sites, insert Eq. (8) into Eq. (21) one gets

$$\begin{aligned} V_k &= \langle k|V = R\langle k|GI_i \\ V_k &= RI_0[G(k,i) - G(k,j)]. \end{aligned} \tag{22}$$

Thus, the resistance between the site r_i and r_j is

$$R(i,j) = \frac{V_i - V_j}{I}. \tag{23}$$

Inserting Eq. (22) into Eq. (23) one obtains

$$R(i,j) = R[G(i,i) + G(j,j) - G(j,i) - G(i,J)]. \tag{24}$$

Now

$$LG = -1 \Rightarrow (L_{01} + L_2)G = -1. \tag{25}$$

Multiplying Eq. (25) by $G_{01} \Rightarrow$

$$G_{01}(L_{01}G + L_2 G) = -G_{01};$$

$$G = G_{01} + G_{01} L_2 G. \tag{26}$$

where $G_{01}(i, j)$ is the LGF due to one broken bond, and it is given in Cserti *et al.* (2002) as:

$$G_{01}(i, j) = \langle i|G_{01}|j\rangle = G_0(i, j) + \frac{[G_0(i, i_0) - G_0(i, j_0)][G_0(i_0, j) - G_0(j_0, j)]}{1 - 2[G_0(i_0, i_0) - G_0(j_0, j_0)]}. \tag{27}$$

Equations (26) and (27) is called Dyson's equation and its solution can be found by the iteration method, and since L_2 has a special form (e.g., Eq. (16)) then one can apply the identity operator (Cserti *et al.*, 2002)

$$(A + |x\rangle\langle y|)^{-1} = A^{-1} - \frac{A^{-1}|x\rangle\langle y|A^{-1}}{1 + \langle y|A^{-1}|x\rangle} \tag{28}$$

From Eq. (25) we have

$$G = -(L_{01} + L_2)^{-1}. \tag{29}$$

Using the above identity with $A = L_{01}$, $|x\rangle = |k_0\rangle - |l_0\rangle$ and $\langle y| = \langle k_0| - \langle l_0|$. Thus,

$$\begin{aligned} G &= -L_{01}^{-1} + \frac{L_{01}^{-1}(|k_0\rangle - |l_0\rangle)(\langle k_0| - \langle l_0|)L_{01}^{-1}}{1 + (\langle k_0| - \langle l_0|)L_{01}^{-1}(|k_0\rangle - |l_0\rangle)}; \\ &= G_{01} + \frac{G_{01}(|k_0\rangle - |l_0\rangle)(\langle k_0| - \langle l_0|)G_{01}}{1 - [G_{01}(k_0, k_0) - G_{01}(k_0, l_0) - G_{01}(l_0, k_0) + G_{01}(l_0, l_0)]}. \end{aligned} \tag{30}$$

The matrix element of G can be expressed with the matrix element of G_{01} as

$$\begin{aligned} G(i, j) = \langle i|G|j\rangle \Rightarrow G(i, j) &= G_{01}(i, j) \\ &+ \frac{[G_{01}(i, k_0) - G_{01}(i, l_0)][G_{01}(k_0, j) - G_{01}(l_0, j)]}{1 - [G_{01}(k_0, k_0) + G_{01}(l_0, l_0) - 2G_{01}(k_0, l_0)]}. \end{aligned} \tag{31}$$

Finally, the resistance between $\vec{r}_i$ and $\vec{r}_j$ can be obtained by using Eqs. (31) and (24)

$$\begin{aligned} \frac{R(i, j)}{R} &= G(i, i) + G(j, j) - G(i, j) - G(j, i) \\ &= G_{01}(i, i) + \frac{[G_{01}(i, k_0) - G_{01}(i, l_0)][G_{01}(k_0, i) - G_{01}(l_0, i)]}{1 - [G_{01}(k_0, k_0) + G_{01}(l_0, l_0) - 2G_{01}(k_0, l_0)]} + G_{01}(j, j) \\ &\quad + \frac{[G_{01}(j, k_0) - G_{01}(j, l_0)][G_{01}(k_0, j) - G_{01}(l_0, j)]}{1 - [G_{01}(k_0, k_0) + G_{01}(l_0, l_0) - 2G_{01}(k_0, l_0)]} - 2G_{01}(j, i) \\ &\quad + \frac{[G_{01}(j, k_0) - G_{01}(j, l_0)][G_{01}(k_0, i) - G_{01}(l_0, i)]}{1 - [G_{01}(k_0, k_0) + G_{01}(l_0, l_0) - 2G_{01}(k_0, l_0)]} \\ &\quad + \frac{[G_{01}(i, k_0) - G_{01}(i, l_0)][G_{01}(k_0, j) - G_{01}(l_0, j)]}{1 - [G_{01}(k_0, k_0) + G_{01}(l_0, l_0) - 2G_{01}(k_0, l_0)]}. \end{aligned} \tag{32}$$

The above equation can be rewritten as

$$\begin{aligned} R(i,j) = {} & R[G_{01}(i,i) + G_{01}(j,j) - 2G_{01}(i,j) \\ & + \frac{1}{1 - [G_{01}(k_0,k_0) + G_{01}(l_0,l_0) - 2G_{01}(k_0,l_0)]} \\ & \times [[G_{01}(i,k_0) - G_{01}(i,l_0)][G_{01}(k_0,i) - G_{01}(l_0,i)] \\ & + [G_{01}(j,k_0) - G_{01}(j,l_0)][G_{01}(k_0,j) - G_{01}(l_0,j)] \\ & - 2[G_{01}(i,k_0) - G_{01}(i,l_0)][G_{01}(k_0,j) - G_{01}(l_0,j)]]]. \end{aligned} \tag{33}$$

The above equation (i.e. Eq. (33)) can be rewritten as:

$$\begin{aligned} R(i,j) = {} & R[G_{01}(i,i) + G_{01}(j,j) - 2G_{01}(i,j) \\ & + \frac{1}{1 - [G_{01}(k_0,k_0) + G_{01}(l_0,l_0) - 2G_{01}(k_0,l_0)]} \\ & \times [[G_{01}(i,k_0) - G_{01}(i,l_0)] - [G_{01}(j,k_0) - G_{01}(j,l_0)]]^2]. \end{aligned} \tag{34}$$

The above equation can be simplified as:

$$\begin{aligned} R(i,j) = R_{01}(i,j) + \frac{R}{1 - \frac{R'_{01}(k_0,l_0)}{R}} \{([G_{01}(i,k_0) - G_{01}(i,l_0)] \\ - [G_{01}(j,k_0) - G_{01}(j,l_0)])\}^2 \end{aligned} \tag{35}$$

where $R_{01}(i,j)$ is the resistance due to removing the bond $(i_0 j_0)$ only and it is given as (Cserti *et al.*, 2002):

$$R_{01}(i,j) = R\left\{R_0(i,j) + \frac{[R_0(i,j_0) + R_0(j,i_0) - R_0(i,i_0) - R_0(j,j_0)]^2}{4[1 - R_0(i_0,j_0)]}\right\}. \tag{36}$$

and $R'_{01}(k_0,l_0)$ is the resistance between the two ends of the removed bond $(k_0 l_0)$ as affected from the removed bond $(i_0 j_0)$. From Eq. (36) we may write:

$$R_{01}(k_0,l_0) = R\left\{R_0(k_0,l_0) + \frac{[R_0(k_0,j_0) + R_0(l_0,i_0) - R_0(k_0,i_0) - R_0(l_0,j_0)]^2}{4[1 - R_0(i_0,j_0)]}\right\}. \tag{37}$$

Inserting Eq. (31) into Eq. (36), one gets some straight forward but lengthy algebra as the following

$$\begin{aligned} R(i,j) = R_{01}(i,j) + \frac{R}{1 - \frac{R'_{01}(k_0,l_0)}{R}} \{G_0(i,k_0) + G_0(j,l_0) - G_0(i,l_0) - G_0(j,k_0) \\ + \frac{1}{1 - 2[G_0(i_0,i_0) - G_0(i_0,j_0)]} (G_0(i,i_0) + G_0(j,j_0) - 2G_0(i,j_0)) \end{aligned}$$

$$\times (G_0(j_0, l_0) + G_0(j_0, k_0) - G_0(i_0, l_0) - G_0(i_0, k_0))\}^2. \tag{38}$$

Inserting Eq. (12) into Eq. (38) and making use of the symmetry of the perfect LGF (i.e. $G_0(n, n) = G_0(m, m)$), one yields

$$R(i, j) = R_{01}(i, j) + \frac{R}{1 - \frac{R'_{01}(k_0, l_0)}{R}} \left\{ \frac{R_0(j, k_0) + R_0(i, l_0) - R_0(j, l_0) - R_0(i, k_0)}{2R} \right.$$
$$+ \frac{1}{1 - \frac{R_0(i_0, j_0)}{R}} \left(\frac{2R_0(i, j_0) - R_0(i, i_0) - R_0(j, j_0)}{2R} \right)$$
$$\left. \times \left(\frac{R_0(i_0, k_0) + R_0(i_0, l_0) - R_0(j_0, l_0) - R_0(j_0, k_0)}{2R} \right) \right\}^2 \tag{39}$$

This is our final result for the resistance between two arbitrary sites $\vec{r}_i$ and $\vec{r}_j$ of the perturbed lattice in which the bonds $(i_0 j_0)$ and $(k_0 l_0)$ are removed.

To check our result, take $k_0 \to 0$ and $l_0 \to 0$ (i.e. reduce the problem to one broken bond), one gets

$$R(i, j) = R_{01}(i, j)$$
$$= R \left\{ R_0(i, j) + \frac{[R_0(i, j_0) + R_0(j, i_0) - R_0(i, i_0) - R_0(j, j_0)]^2}{4[1 - R_0(i_0, j_0)]} \right\} \tag{40}$$

which is the result obtained in Cserti *et al.* (2002) due to removing the bond $(i_0 j_0)$ alone.

Our final form for the resistance in the perturbed lattice is valid for any lattice structure in which each cell has only one lattice site. This is due to the fact that the explicit form of the lattice Laplacian defined above was not used in the derivation of Eq. (39).

Again, for large separation between the sites i and j the resistance in an infinite perturbed square lattice goes to infinity.

4. NUMERICAL RESULTS

In this section, numerical results are presented for an infinite square lattice including both the perfect and perturbed cases. The resistance between the origin and the lattice site *(l, m)* is calculated in Asad *et al.* (2004) using the so-called recurrence formulae for the resistance of an infinite square lattice presented in Cserti (2000).

On the perturbed square lattice—where two bonds are broken—the resistance can be calculated from Equation (39). In this work, the site $\vec{r}_i$ is fixed while the site $\vec{r}_j$ is moved along the line of the removed bond. Here we considered three cases; first, when the first removed bond is between $i_0 = (0, 0)$ and $j_0 = (1, 0)$, whereas the second broken bond is between $k_0 = (1, 0)$ and $l_0 = (2, 0)$. Our calculated

Table I. Calculated Values for the Resistance of an Infinite Square Lattice Between the Origin and the Site $j = (j_x, 0)$, for a Perfect Lattice ($R_0(i, j)/R$); Perturbed Lattice due to the Broken Bond Between (0, 0) and (1, 0) -($R_{01}(i, j)/R$)-; Perturbed Lattice due to Removing the Bonds Between (0, 0), (1, 0) and (1, 0), (2, 0) -($R_1(i, j)/R$)-; Perturbed Lattice due to Removing the Bonds Between (0, 0), (1, 0) and (2, 0), (3, 0) -($R_2(i, j)/R$)- and Finally, Perturbed Lattice due to Removing the Bonds Between (1, 0), (2, 0) and (2, 0), (3, 0) -($R_3(i, j)/R$)-

$j = (j_x, 0)$	$R_0(i, j)/R$	$R_{01}(i, j)/R$	$R_1(i, j)/R$	$R_2(i, j)/R$	$R_3(i, j)/R$
(0, 0)	0	0	0	0	0
(1, 0)	0.5	1	1.11114	1.03612	1.03026
(2, 0)	0.72676	0.99085	1.633	1.00781	1.04855
(3, 0)	0.86056	1.06142	1.29119	1.33293	1.56294
(4, 0)	0.95399	1.13006	1.24534	1.21548	1.2792
(5, 0)	1.0258	1.18929	1.25686	1.23009	1.24999
(6, 0)	1.08423	1.24015	1.28246	1.26407	1.26797
(7, 0)	1.13352	1.28438	1.31149	1.29979	1.29704
(8, 0)	1.17616	1.32339	1.34071	1.33376	1.3285
(9, 0)	1.21375	1.35825	1.36905	1.36535	1.36985
(10, 0)	1.24735	1.38971	1.39613	1.39458	1.38983
(−1, 0)	0.5	0.53733	0.58326	0.54726	0.54363
(−2, 0)	0.72676	0.79381	0.82357	0.80025	0.79586
(−3, 0)	0.86056	0.94322	0.9624	0.94751	0.9435
(−4, 0)	0.95399	1.04566	1.05775	1.0485	1.04572
(−5, 0)	1.0258	1.12329	1.13059	1.12513	1.12414
(−6, 0)	1.08423	1.18580	1.1899	1.18693	1.18812
(−7, 0)	1.13352	1.23811	1.24013	1.23876	1.24238
(−8, 0)	1.17616	1.28307	1.28384	1.28339	1.28962
(−9, 0)	1.21375	1.32251	1.32266	1.32263	1.33158
(−10, 0)	1.24735	1.35762	1.35762	1.35764	1.36937

values of the resistance (i.e. $R_1(i, j)/R$) are arranged in Table I. In the second case, the first broken bond is taken between $i_0 = (0, 0)$ and $j_0 = (1, 0)$, whereas the second broken bond is between $k_0 = (2, 0)$ and $l_0 = (3, 0)$. Our results (i.e. $R_2(i, j)/R$) are arranged in Table I. Finally, we consider the case where the first removed bond is between $i_0 = (1, 0)$ and $j_0 = (2, 0)$, whereas the second broken bond is between $k_0 = (2, 0)$ and $l_0 = (3, 0)$. Again our calculated values of the resistance (i.e. $R_3(i, j)/R$) are arranged in Table I below.

In Figs. 1–3 the resistance for the perfect and the above three perturbed cases are plotted as functions of j_x. While, in Fig. 4 the resistance is plotted as functions of j_x for the perturbed lattices (i.e., case one and case three above).

One can see that the resistance when two bonds are broken is always larger than that when one bond is broken. This is due to the positivity of the second term in Eq. (39). This also means that the resistance when two bonds are broken is always larger than the perfect resistance, and in general, one can say that; as

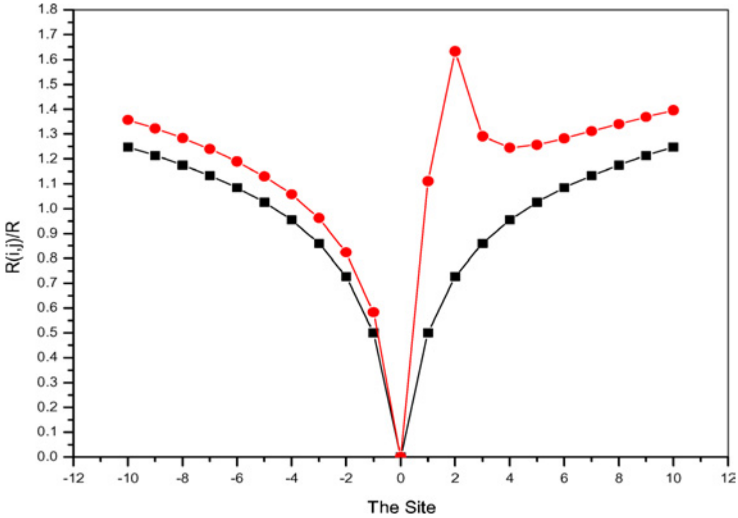

Fig. 2. The resistance between $i = (0, 0)$ and $j = (j_x, 0)$ along the [10] direction of the perfect (squares) and the perturbed (circles) square lattice as a function of j_x. The ends of the removed bonds are $i_0 = (0, 0)$ and $j_0 = (1, 0)$, $k_0 = (1, 0)$ and $l_0 = (2, 0)$.

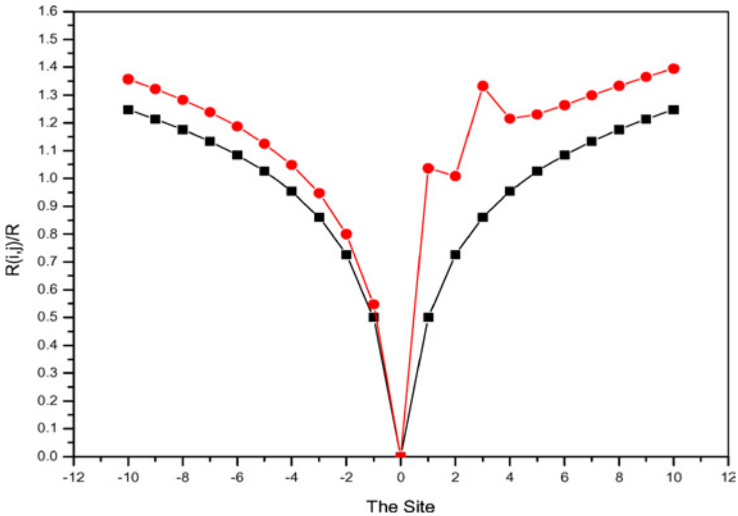

Fig. 3. The resistance between $i = (0, 0)$ and $j = (j_x, 0)$ along the [10] direction of the perfect (squares) and the perturbed (circles) square lattice as a function of j_x. The ends of the removed bonds are $i_0 = (0, 0)$ and $j_0 = (1, 0)$, $k_0 = (2, 0)$ and $l_0 = (3, 0)$.

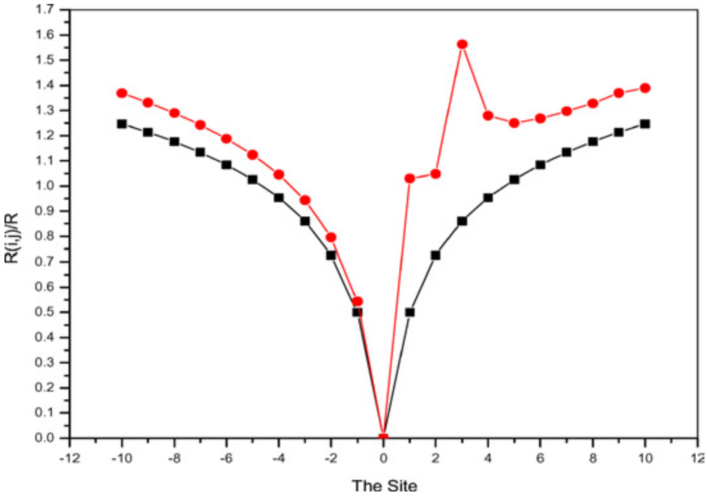

Fig. 4. The resistance between $i = (0, 0)$ and $j = (j_x, 0)$ along the [10] direction of the perfect (squares) and the perturbed (circles) square lattice as a function of j_x. The ends of the removed bonds are $i_0 = (1, 0)$ and $j_0 = (2, 0)$, $k_0 = (2, 0)$ and $l_0 = (3, 0)$.

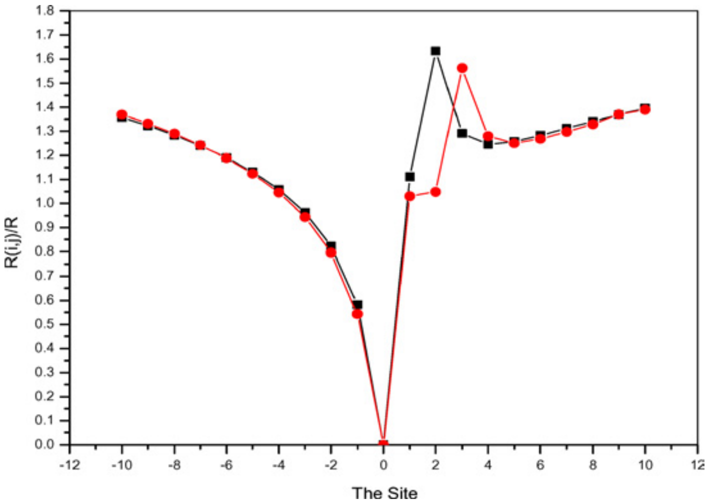

Fig. 5. The resistance between $i = (0, 0)$ and $j = (j_x, 0)$ along the [10] direction of the perturbed (squares) and the shifted perturbed (circles) square lattice as a function of j_x. The ends of the removed bonds for the perturbed are $i_0 = (0, 0)$ and $j_0 = (1, 0)$, $k_0 = (1, 0)$ and $l_0 = (2, 0)$ while for the shifted perturbed are $i_0 = (1, 0)$ and $j_0 = (2, 0)$, $k_0 = (2, 0)$ and $l_0 = (3, 0)$.

the number of broken bonds increases in an infinite square lattice the perturbed resistance increases.

Finally, from Figs. 1–4 one can see that the resistance in a perturbed infinite square lattice is not symmetric under the transformation $j_x \to -j_x$. This is due to the fact that the inversion symmetry of the lattice has been broken. It can also be seen from the figures that increasing the distance between the sites $\vec{r}_i$ and $\vec{r}_j$ the resistance tends to that of the perfect lattice. (Fig. 5).

REFERENCES

Asad, J. H., Sakaji, A., Hijjawi, R. S., and Khalifeh, J. M. (2004). On the Resistance of an Infinite Square Network of Identical Resistors (*Theoretical and Experimental Comparison*). *Hadronic Journal* (accepted).

Atkinson, D. and Van Steenwijk, F. J. (1999). *American Journal of Physics* **67**, 486.

Bartis, F. J. (1967). *American Journal of Physics* **35**, 354.

Cserti, J. (2000). *American Journal of Physics* **68**, 896.

Cserti, J., David, G., and Piroth, A. (2002). *American Journal of Physics* **70**, 153.

Doyle, P. G. and Snell, J. L. (1984). Random walks and Electric Networks, The Carus Mathematical Monograph, Series **22**, The Mathematical Association of America, USA, 83 pp.

Economou, E. N. (1983). *Green's Functions in Quantum Physics*, Spriger-Verlag, Berlin.

Hijjawi, R. (2002). PhD Thesis, University of Jordan (unpublished).

Katsura, S. and Inawashiro, S. (1971). *Journal of Mathematical Physics* **12**(8), 1622.

Kirchhoff, G. (1847). *Annals of Physik und Chemie* **72**, 497.

Lovăsz, L. (1996). Random walks on graphs: A survey, in Combinatorics, Paul Erdös is Eighty, Vol. 2. In D. Miklós, V. T. Sós, and T. Szónyi, eds., János Bolyai Mathematical Society, Budepest, pp. 353–398. Also at http://research.microsoft.com∼Lovăsz/as a survey paper.

Monwhea, J. (2000). *American Journal of Physics* **68**(1), 37.

Morita, T. (1971). *Journal of Mathematical Physics* **12**(8), 1744.

Morita, T. and Horiguchi, T. (1971). *Journal of Mathematical Physics* **12**(6), 986.

Morita, T. and Horiguchi, T. (1972). *Journal of Physics A: Mathematics and General* **5**(1), 67.

Redner, S. (2001). *A Guide to First-passage Processes*, Cambridge Press, Cambridge, UK.

Van der Pol, B. and Bremmer, H. (1955). *Operational Calculus Based on the Two-Sided Laplace Integral*, Cambridge University Press, England.

Venezian, G. (1994). *American Journal of Physics* **62**, 1000.

Wu, F. Y. (2004). *Journal of Physics A: Mathematics and General* (37), 6653.